An upper bound on the fractional chromatic number of triangle-free subcubic graphs

Chun-Hung Liu

Georgia Institute of Technology

May 10, 2012

Definition

An $(a : b)$-coloring of a graph G is a function f which maps the vertices of G into b-element subsets of some set of size a in such a way that $f(u)$ is disjoint from $f(v)$ for every two adjacent vertices u and v in G.

Definition

An $(a:b)$-coloring of a graph G is a function f which maps the vertices of G into b-element subsets of some set of size a in such a way that $f(u)$ is disjoint from $f(v)$ for every two adjacent vertices u and v in G.
Every $(a:1)$-coloring is a proper vertex a-coloring.

Definition

An $(a : b)$-coloring of a graph G is a function f which maps the vertices of G into b-element subsets of some set of size a in such a way that $f(u)$ is disjoint from $f(v)$ for every two adjacent vertices u and v in G.

Every $(a : 1)$-coloring is a proper vertex a-coloring.

The *fractional chromatic number* $\chi_f(G)$ is the infimum of a/b over all pairs of positive integers a, b such that G has an $(a : b)$-coloring.

Definition

An $(a:b)$-coloring of a graph G is a function f which maps the vertices of G into b-element subsets of some set of size a in such a way that $f(u)$ is disjoint from $f(v)$ for every two adjacent vertices u and v in G.
Every $(a:1)$-coloring is a proper vertex a-coloring.
The *fractional chromatic number* $\chi_f(G)$ is the infimum of a/b over all pairs of positive integers a, b such that G has an $(a:b)$-coloring.
So $\chi_f(G) \leq \chi(G)$.

Definition

An $(a:b)$-coloring of a graph G is a function f which maps the vertices of G into b-element subsets of some set of size a in such a way that $f(u)$ is disjoint from $f(v)$ for every two adjacent vertices u and v in G.
Every $(a:1)$-coloring is a proper vertex a-coloring.
The *fractional chromatic number* $\chi_f(G)$ is the infimum of a/b over all pairs of positive integers a, b such that G has an $(a:b)$-coloring.
So $\chi_f(G) \leq \chi(G)$.
In fact, the infimum is the minimum.

A conjecture

If G has an $(a:b)$-coloring, then $\alpha(G) \geq bn/a$.

A conjecture

If G has an $(a:b)$-coloring, then $\alpha(G) \geq bn/a$.
Staton (1979) proved that $\alpha(G) \geq 5n/14$ for every triangle-free subcubic graph (i.e. graph of max. degree ≤ 3).

A conjecture

If G has an $(a:b)$-coloring, then $\alpha(G) \geq bn/a$.
Staton (1979) proved that $\alpha(G) \geq 5n/14$ for every triangle-free subcubic graph (i.e. graph of max. degree ≤ 3).
Fajtlowicz (1978) pointed out that the generalized Petersen graph $P(7,2)$ has 14 vertices and no independent set of size 6.

A conjecture

If G has an $(a : b)$-coloring, then $\alpha(G) \geq bn/a$.
Staton (1979) proved that $\alpha(G) \geq 5n/14$ for every triangle-free subcubic graph (i.e. graph of max. degree ≤ 3).
Fajtlowicz (1978) pointed out that the generalized Petersen graph $P(7,2)$ has 14 vertices and no independent set of size 6.
Heckman and Thomas (2001) gave a short proof of Staton's Theorem, and they gave the following conjecture.
Conjecture: *The fractional chromatic number of every triangle-free subcubic graph is at most* $14/5$.

A conjecture

If G has an $(a:b)$-coloring, then $\alpha(G) \geq bn/a$.
Staton (1979) proved that $\alpha(G) \geq 5n/14$ for every triangle-free subcubic graph (i.e. graph of max. degree ≤ 3).
Fajtlowicz (1978) pointed out that the generalized Petersen graph $P(7,2)$ has 14 vertices and no independent set of size 6.
Heckman and Thomas (2001) gave a short proof of Staton's Theorem, and they gave the following conjecture.
Conjecture: *The fractional chromatic number of every triangle-free subcubic graph is at most* $14/5$.
The statement that $\chi_f(G) \leq 14/5$ is equivalent to the statement that for every weight function defined on $V(G)$, there is an independent set I of G such that the sum of weights of vertices in I is at least $5w(G)/14$.

Known results

Let G be a triangle-free subcubic graph.

1. (Hatami and Zhu 2009) $\chi_f(G) \leq 3 - \frac{3}{64} \approx 2.953$, and $\chi_f(G) \leq 2.78571$ if the girth of G is at least 7.

Known results

Let G be a triangle-free subcubic graph.

1. (Hatami and Zhu 2009) $\chi_f(G) \leq 3 - \frac{3}{64} \approx 2.953$, and $\chi_f(G) \leq 2.78571$ if the girth of G is at least 7.
2. (Lu and Peng 2010^+) $\chi_f(G) \leq 3 - \frac{3}{43} \approx 2.930$.

Known results

Let G be a triangle-free subcubic graph.

1. (Hatami and Zhu 2009) $\chi_f(G) \leq 3 - \frac{3}{64} \approx 2.953$, and $\chi_f(G) \leq 2.78571$ if the girth of G is at least 7.
2. (Lu and Peng 2010^+) $\chi_f(G) \leq 3 - \frac{3}{43} \approx 2.930$.
3. (Ferguson, Kaiser, and Král 2012^+) $\chi_f(G) \leq 32/11 \approx 2.909$.

Known results

Let G be a triangle-free subcubic graph.

1. (Hatami and Zhu 2009) $\chi_f(G) \leq 3 - \frac{3}{64} \approx 2.953$, and $\chi_f(G) \leq 2.78571$ if the girth of G is at least 7.
2. (Lu and Peng 2010$^+$) $\chi_f(G) \leq 3 - \frac{3}{43} \approx 2.930$.
3. (Ferguson, Kaiser, and Král 2012$^+$) $\chi_f(G) \leq 32/11 \approx 2.909$.

Theorem (L.)

The fractional chromatic number of every triangle-free subcubic graph is at most $43/15 \approx 2.867$.

Larger maximum degree

Let G be a K_Δ-free graph of maximum degree Δ other than C_8^2 and $C_5 \boxtimes K_2$.

1. (King, Lu, and Peng 2012) $\chi_f(G) \leq 4 - \frac{2}{67} \approx 3.9701$ when $\Delta = 4$. $(\chi_f(C_{11}^2) = 4 - \frac{1}{3} \approx 3.6778)$.
2. (King, Lu, and Peng 2012) $\chi_f(G) \leq 5 - \frac{2}{67} \approx 4.9701$ when $\Delta = 5$. $(\chi_f(C_7 \boxtimes K_2) = 5 - \frac{1}{3} \approx 4.6778)$.
3. (Edwards and King) $\chi_f(G) \leq 6 - \frac{153}{3431} \approx 5.9554$ when $\Delta = 6$. $(\chi_f((C_5 \boxtimes K_3) - 4v) = 6 - \frac{1}{2} = 5.5)$.
4. (Edwards and King) $\chi_f(G) \leq 7 - \frac{80}{889} \approx 6.9100$ when $\Delta = 7$. $(\chi_f((C_5 \boxtimes K_3) - 2v) = 7 - \frac{1}{2} = 6.5)$.
5. (Edwards and King) $\chi_f(G) \leq 8 - \frac{17280}{152209} \approx 7.8864$ when $\Delta = 8$. $(\chi_f(C_5 \boxtimes K_3) = 8 - \frac{1}{2} = 7.5)$.
6. (Edwards and King) $\chi_f(G) \leq 9 - \frac{17}{130} \approx 8.8692$ when $\Delta = 9$.
7. (Edwards and King) $\chi_f(G) \leq 10 - \frac{8565625}{60177971} \approx 9.8577$ when $\Delta = 10$.

Main ideas

Find a "good" proper 3-coloring f of G such that for each $1 \leq i < j \leq 3$, the subgraph, denoted by $G^{(i,j)}$, of G induced by vertices of color i and j has some "good" structures.

Main ideas

Find a "good" proper 3-coloring f of G such that for each $1 \leq i < j \leq 3$, the subgraph, denoted by $G^{(i,j)}$, of G induced by vertices of color i and j has some "good" structures.
Define two maps $g_1^{(i,j)}$ and $g_2^{(i,j)}$ from $V(G)$ to subsets of [14] such that adjacent vertices receive disjoint sets, and

$$\begin{aligned}&|g_1^{(i,j)}(v)| + |g_2^{(i,j)}(v)| \\ &= \begin{cases} 4, & \text{if } f(v) \notin \{i,j\}; \\ 16 - 2\deg_{G^{(i,j)}}(v) - 2 \cdot 1_{I^{(i,j)}}(v), & \text{if } f(v) \in \{i,j\}. \end{cases}\end{aligned}$$

for some independent set $I^{(i,j)} \subseteq \{v : \deg_{G^{(i,j)}}(v) = 3\}$ of G.

Main ideas

Define $g^{(i,j)}$ by the map from $V(G)$ to subsets of [28] such that $g^{(i,j)}(v) = g_1^{(i,j)}(v) \cup (g_2^{(i,j)}(v) + 14)$ for every $1 \leq i < j \leq 3$ and vertex v.

Main ideas

Define $g^{(i,j)}$ by the map from $V(G)$ to subsets of [28] such that $g^{(i,j)}(v) = g_1^{(i,j)}(v) \cup (g_2^{(i,j)}(v) + 14)$ for every $1 \leq i < j \leq 3$ and vertex v.

Define g by the map from $V(G)$ to subsets of [84] such that $g(v) = g^{(1,2)}(v) \cup (g^{(1,3)} + 28) \cup (g^{(2,3)}(v) + 56)$ for every vertex v.

Main ideas

Define $g^{(i,j)}$ by the map from $V(G)$ to subsets of [28] such that $g^{(i,j)}(v) = g_1^{(i,j)}(v) \cup (g_2^{(i,j)}(v) + 14)$ for every $1 \leq i < j \leq 3$ and vertex v.
Define g by the map from $V(G)$ to subsets of [84] such that $g(v) = g^{(1,2)}(v) \cup (g^{(1,3)} + 28) \cup (g^{(2,3)}(v) + 56)$ for every vertex v.
So for vertex v with $f(v) = 1$,

$$
\begin{aligned}
&|g(v)| \\
&\geq 4 + 32 - 2(\deg_{G^{(1,2)}}(v) + \deg_{G^{(1,3)}}(v)) - 2(1_{I^{(1,2)}}(v) + 1_{I^{(1,3)}}(v)) \\
&= 36 - 2\deg_G(v) - 2(1_{I^{(1,2)}}(v) + 1_{I^{(1,3)}}(v) + 1_{I^{(2,3)}}(v)) \\
&\geq 30 - 2(1_{I^{(1,2)}}(v) + 1_{I^{(1,3)}}(v) + 1_{I^{(2,3)}}(v)).
\end{aligned}
$$

Main ideas

Define $g^{(i,j)}$ by the map from $V(G)$ to subsets of $[28]$ such that $g^{(i,j)}(v) = g_1^{(i,j)}(v) \cup (g_2^{(i,j)}(v) + 14)$ for every $1 \leq i < j \leq 3$ and vertex v.
Define g by the map from $V(G)$ to subsets of $[84]$ such that $g(v) = g^{(1,2)}(v) \cup (g^{(1,3)} + 28) \cup (g^{(2,3)}(v) + 56)$ for every vertex v.
So for vertex v with $f(v) = 1$,

$$
\begin{aligned}
&|g(v)| \\
&\geq 4 + 32 - 2(\deg_{G^{(1,2)}}(v) + \deg_{G^{(1,3)}}(v)) - 2(1_{I^{(1,2)}}(v) + 1_{I^{(1,3)}}(v)) \\
&= 36 - 2\deg_G(v) - 2(1_{I^{(1,2)}}(v) + 1_{I^{(1,3)}}(v) + 1_{I^{(2,3)}}(v)) \\
&\geq 30 - 2(1_{I^{(1,2)}}(v) + 1_{I^{(1,3)}}(v) + 1_{I^{(2,3)}}(v)).
\end{aligned}
$$

Similarly, $|g(v)| \geq 30 - 2(1_{I^{(1,2)}}(v) + 1_{I^{(1,3)}}(v) + 1_{I^{(2,3)}}(v))$ for every vertex v. So g is "almost" a $(84:30)$-coloring.

Main ideas

Define $g^{(i,j)}$ by the map from $V(G)$ to subsets of [28] such that $g^{(i,j)}(v) = g_1^{(i,j)}(v) \cup (g_2^{(i,j)}(v) + 14)$ for every $1 \leq i < j \leq 3$ and vertex v.
Define g by the map from $V(G)$ to subsets of [84] such that $g(v) = g^{(1,2)}(v) \cup (g^{(1,3)} + 28) \cup (g^{(2,3)}(v) + 56)$ for every vertex v.
So for vertex v with $f(v) = 1$,

$$
\begin{aligned}
&|g(v)| \\
&\geq 4 + 32 - 2(\deg_{G^{(1,2)}}(v) + \deg_{G^{(1,3)}}(v)) - 2(1_{I^{(1,2)}}(v) + 1_{I^{(1,3)}}(v)) \\
&= 36 - 2\deg_G(v) - 2(1_{I^{(1,2)}}(v) + 1_{I^{(1,3)}}(v) + 1_{I^{(2,3)}}(v)) \\
&\geq 30 - 2(1_{I^{(1,2)}}(v) + 1_{I^{(1,3)}}(v) + 1_{I^{(2,3)}}(v)).
\end{aligned}
$$

Similarly, $|g(v)| \geq 30 - 2(1_{I^{(1,2)}}(v) + 1_{I^{(1,3)}}(v) + 1_{I^{(2,3)}}(v))$ for every vertex v. So g is "almost" a $(84 : 30)$-coloring.
A minor modification yields an $(86 : 30)$-coloring.

Main ideas

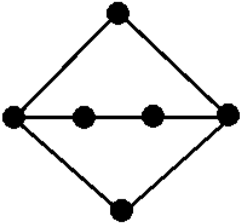
$I^{(1,2)}$, $I^{(1,3)}$, and $I^{(2,3)}$ are pairwise disjoint independent sets, and the union of them is also an independent set.

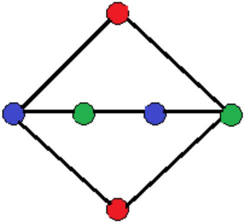
Main ideas

$I^{(1,2)}$, $I^{(1,3)}$, and $I^{(2,3)}$ are pairwisely disjoint independent sets, and the union of them is also an independent set.
Let $I = I^{(1,2)} \cup I^{(1,3)} \cup I^{(2,3)}$.
Define g' by $g'(v) = g(v)$ if $v \notin I$, and $g'(v) \cup \{85, 86\}$ if $v \in I$.
Hence, g' is a (86 : 30)-coloring, so $\chi_f(G) \leq 86/30 = 43/15$.

Graph L_0

Let L_0 be the graph that is obtained by identifying two paths of order four of two different C_5's. A L_0 is a *rainbow* L_0 with respect to a proper 3-coloring f if the shared 4-path is bicolored by f.





Good 3-coloring

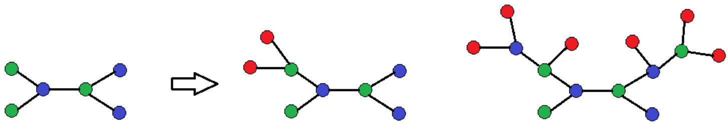
A proper 3-coloring f of a subcubic graph G is *good* if for every $1 \leq i < j \leq 3$:

- G contains no rainbow L_0's with respect to f.

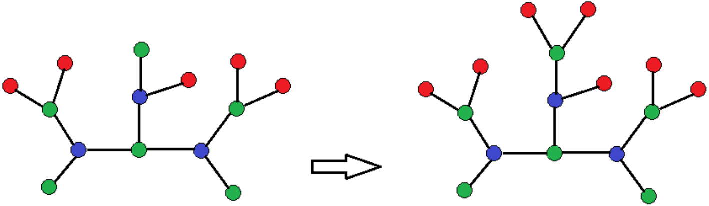
Good 3-coloring

A proper 3-coloring f of a subcubic graph G is *good* if for every $1 \leq i < j \leq 3$:

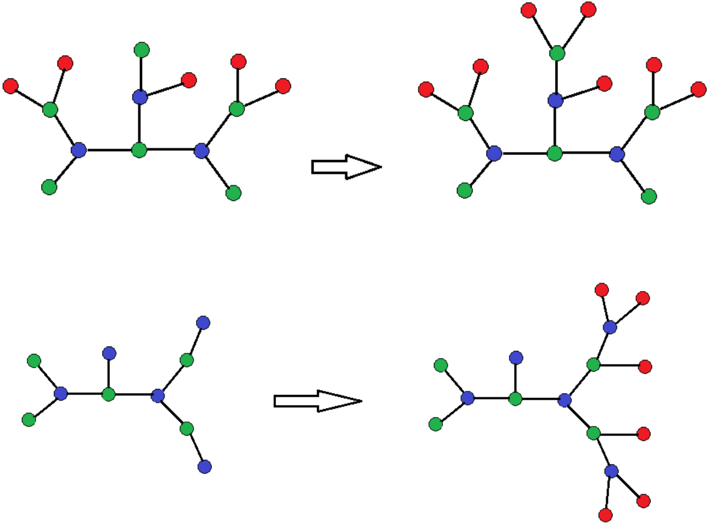
- G contains no rainbow L_0's with respect to f.



Good 3-coloring



Good 3-coloring



Fractionally critical graph

G is *fractionally t-critical* if $\chi_f(G) > t$ but $\chi_f(H) \leq t$ for every proper subgraph H of G.

Lemma

If G is a fractionally t-critical triangle-free subcubic graph, where $t \geq 8/3$, then G has a good 3-coloring.

Next step

We have a good 3-coloring h, and now we want to define $g_1^{(i,j)}$ and $g_2^{(i,j)}$ from $V(G)$ to subsets of [14] such that adjacent vertices receive disjoint sets, and

$$|g_1^{(i,j)}(v)| + |g_2^{(i,j)}(v)| = \begin{cases} 4, & \text{if } h(v) \notin \{i,j\}; \\ 16 - 2\deg_{G^{(i,j)}}(v) - 2 \cdot 1_{I^{(i,j)}}(v), & \text{if } h(v) \in \{i,j\}. \end{cases}$$

Next step

We have a good 3-coloring h, and now we want to define $g_1^{(i,j)}$ and $g_2^{(i,j)}$ from $V(G)$ to subsets of [14] such that adjacent vertices receive disjoint sets, and

$$|g_1^{(i,j)}(v)| + |g_2^{(i,j)}(v)| = \begin{cases} 4, & \text{if } h(v) \notin \{i,j\}; \\ 16 - 2\deg_{G^{(i,j)}}(v) - 2 \cdot 1_{I^{(i,j)}}(v), & \text{if } h(v) \in \{i,j\}. \end{cases}$$

We first define each $g_1^{(i,j)}(v)$ and $g_2^{(i,j)}(v)$ by 2 colors on vertices v with $h(v) \notin \{i,j\}$, and then extend the coloring to $G^{(2,3)}$.

Remark

We can prove that $\chi_f(G) \leq 14/5$ if one can show that every fractionally $14/5$-critical triangle-free subcubic graph has a good 3-coloring such that each two color classes induced a subgraph of maximum degree at most two.

THANK YOU

HAPPY BIRTHDAY, ROBIN!